Five-minute review

1. Does r = .98 prove that increasing x causes y to increase?

2. $\hat{y} = 8 + 2x$. At x = 6, y = 17. Find the residual.

Write your answers before we discuss them.

Sample spaces and probability models

What we will learn

List outcomes, define events and calculate theoretical and experimental probabilities.

Why does it matter?

Before computing probability, decide which outcomes are equally likely. A fair four-sector spinner is a simple model.

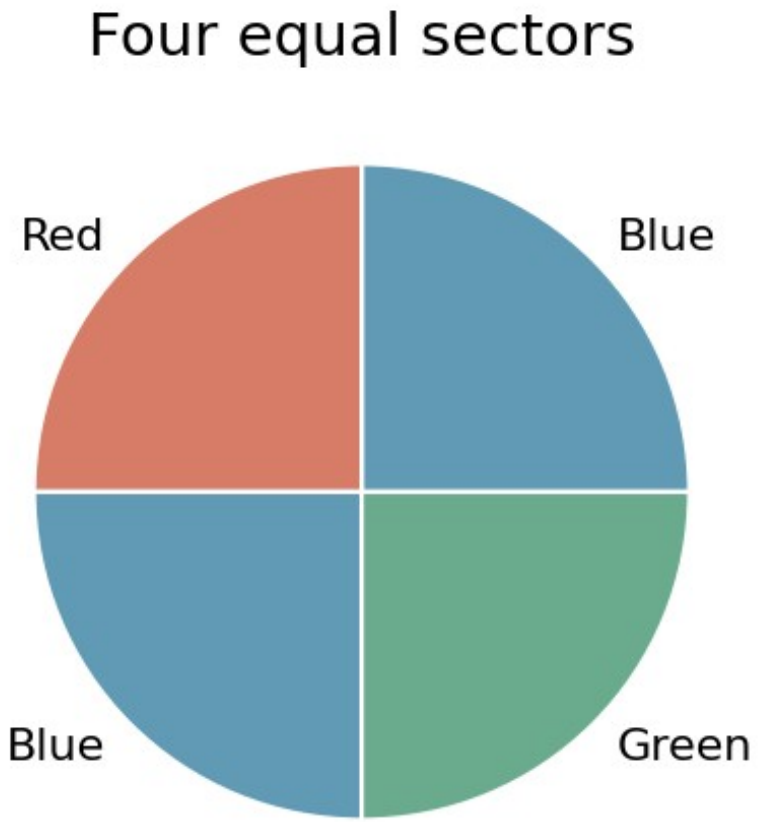
Definitions: Sample spaces and probability models

Random experiment: a process whose outcome is uncertain.

Sample space S: the set of possible outcomes.

Event A: a subset of the sample space; probability P(A) measures its likelihood.

Illustration and interpretation



Notice the pattern

Before computing probability, decide which outcomes are equally likely. A fair four-sector spinner is a simple model.

Illustrative data / model

Formulas

$$P(A) = \frac{|A|}{|S|}$$

$$\widehat{P}(A) = \frac{\text{Observed successes}}{\text{Number of trials}}$$

Important ideas and notation

An experiment produces an outcome; S is the sample space; an event is a subset of S. For equally likely outcomes, P(A)=number in A/number in S.

Experimental probability = observed successes/trials. Subjective probability reflects informed judgment; distinguish it from a mathematical model.

$0 \le P(A) \le 1$; $P(S)=1$ and $P(\varnothing)=0$. A model needs its assumptions stated.

Example 3.1.1

Roll a fair six-sided die. A={2,4,6}. Find the probability of A and describe the event.

Solution:

$P(A)=\frac{3}{6}=\frac{1}{2}$. The event is even.

Example 3.1.2

A spinner lands blue 17 times in 50 trials. Find the experimental probability of blue.

Solution:

Experimental probability .34. This need not equal theoretical probability.

Example 3.1.3 • A fair spinner

What is P(not blue)?

Sector	Color
1	Blue
2	Red
3	Blue
4	Green

Illustrative classroom dataset

Example 3.1.3 • Solution

What is P(not blue)?

Solution:

$\frac{2}{4} = \frac{1}{2}$.

Individual practice

1. List outcomes of two ordered coin tosses.

2. For a fair die, P(number >4).

3. Can 1.08 be a probability?

Group activity

Make a spinner with four equal sectors, two blue, one red, one green. Predict blue probability and compare with 20 trials.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. HH, HT, TH, TT.

2. $\frac{2}{6} = \frac{1}{3}$.

3. No.

Group solution

Solution:

Theoretical .50

experimental count/20 varies.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. When can favorable outcomes/total outcomes be used directly?

2. Roll a fair die once. Find the probability of a number greater than 4.

Show your reasoning. Submit before leaving.

Five-minute review

1. When can favorable outcomes/total outcomes be used directly?

2. List outcomes of two ordered coin tosses.

Write your answers before we discuss them.

Simulation and the law of large numbers

What we will learn

Use random-number simulations and explain long-run relative frequency.

Why does it matter?

Small experiments can give surprising results. The law of large numbers describes long-run behavior, not a guarantee for every run.

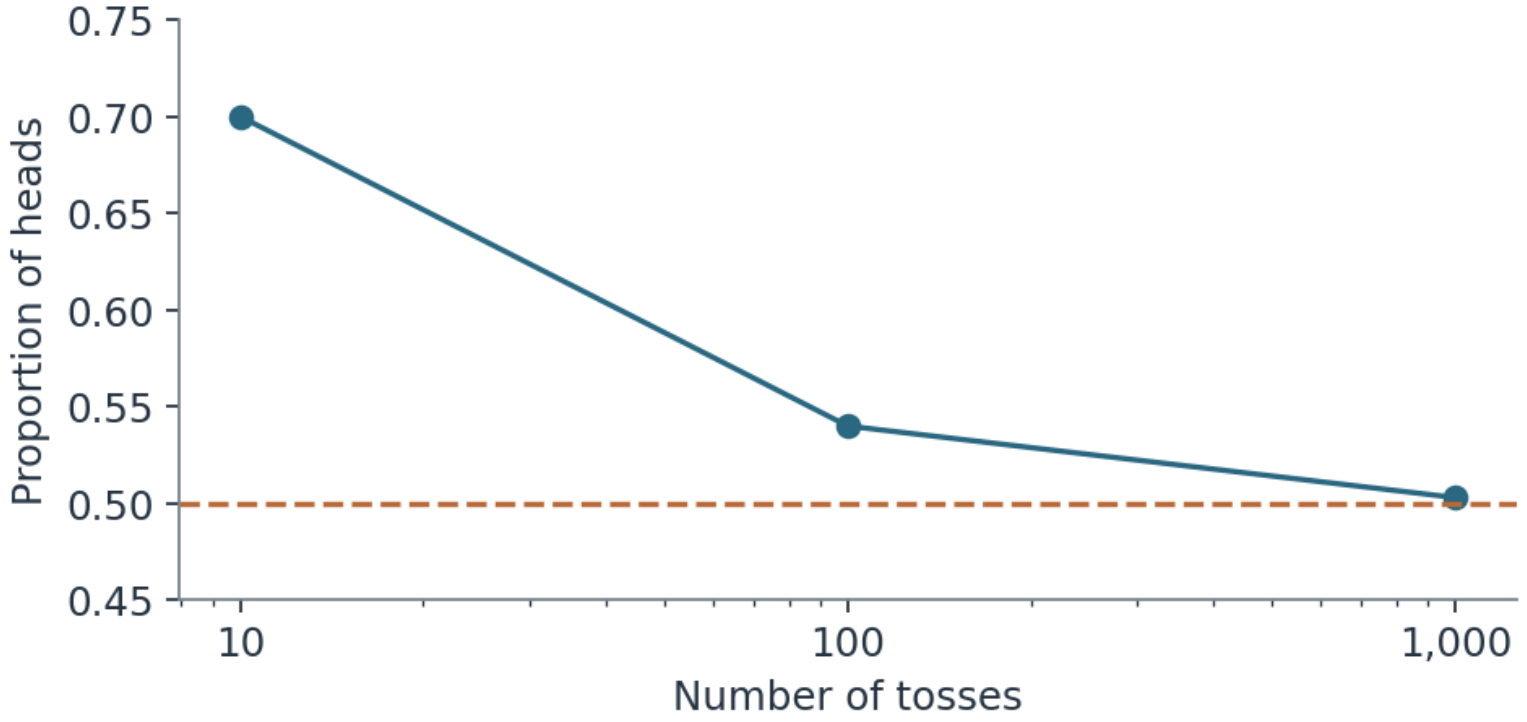
Definitions: Simulation and the law of large numbers

Experimental probability: observed successes divided by number of trials.

Simulation: a model that imitates a random process.

Law of large numbers: under independent identical trials, the relative frequency approaches the true probability as trials increase.

Illustration and interpretation



Notice the pattern

Small experiments can give surprising results. The law of large numbers describes long-run behavior, not a guarantee for every run.

Illustrative data / model

Formulas

$$\widehat{P}(A) = \frac{\text{Successes}}{\text{Trials}}$$

Important ideas and notation

Simulation uses random numbers to imitate a chance process. Assign equally likely numbers to outcomes in proportion to modeled probabilities.

Law of large numbers: as independent identical trials increase, relative frequency tends to approach the true probability. It does not guarantee short-run balance.

Using the method correctly

For a fair choice between six groups, generate random integers 1–6. Spreadsheet RANDBETWEEN(1,6) or R sample(1:6,100,replace=TRUE) works.

Example 3.2.1

A fair coin simulation gives 7 heads in 10 trials, then 503 heads in 1000.

Solution:

Relative frequencies .70 and .503. The second is closer to .50, though this is not guaranteed in every pair of runs.

Example 3.2.2

Simulate a 30% event using digits 0–9.

Solution:

Assign 0,1,2 to success
others to failure.

Example 3.2.3 • Coin-toss results

Which result is closest to the theoretical probability?

Tosses	Heads	Proportion
10	7	0.700
100	54	0.540
1000	503	0.503

Illustrative classroom dataset

Example 3.2.3 • Solution

Which result is closest to the theoretical probability?

Solution:

$$\frac{503}{1000}$$

absolute difference from 0.5 is 0.003.

Individual practice

1. A die rolls 6 in 14 of 60 trials.

2. After 5 tails, next fair toss P(heads)?

3. Which is generally more stable: 20 or 2000 trials?

Group activity

Run 30 then 300 simulated die rolls and compare proportion of sixes to 1/6; record seed if supported.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. Experimental probability $\frac{14}{60} \approx .233$.

2. .50, independence.

3. 2000.

Group solution

Solution:

Results vary

compare absolute deviations from .1667, not a guarantee of monotonic improvement.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Does the law of large numbers mean outcomes must alternate?

2. A fair coin is tossed 200 times and lands heads 108 times. Find experimental probability and compare with theory.

Show your reasoning. Submit before leaving.

Five-minute review

1. Does the law of large numbers mean outcomes must alternate?

2. A die rolls 6 in 14 of 60 trials.

Write your answers before we discuss them.

Complements, unions and Venn diagrams

What we will learn

Use OR, AND, NOT and the addition rule.

Why does it matter?

When students can choose more than one activity, adding both totals counts the overlap twice.

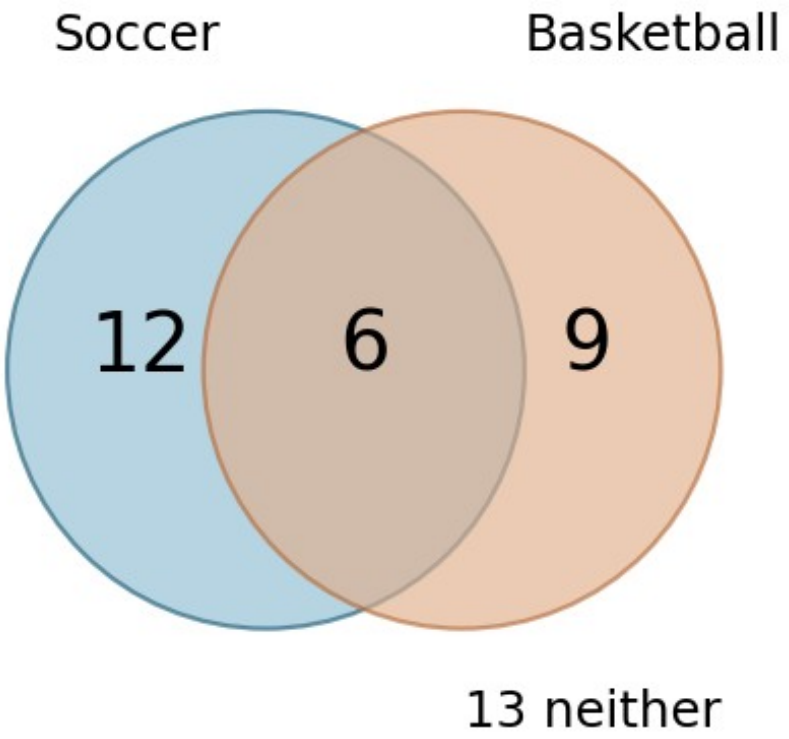
Definitions: Complements unions and venn diagrams

Complement A^c: the outcomes not in A.

Union $A \cup B$: outcomes in A, B or both; intersection $A \cap B$: outcomes in both.

Mutually exclusive events: events with no shared outcomes.

Illustration and interpretation



Notice the pattern

When students can choose more than one activity, adding both totals counts the overlap twice.

Illustrative data / model

Formulas

$$P(A^c) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Important ideas and notation

$A \cup B$ = A or B (inclusive); $A \cap B$ = A and B; A^c = not A. $P(A^c) = 1 - P(A)$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Disjoint events have no shared outcomes.

Venn diagrams partition the sample space; use intersection first, then A-only, B-only and neither.

Example 3.3.1

40 students: 18 soccer, 15 basketball, 6 both. Find the probability that a student likes at least one sport and the probability of neither.

Solution:

Union count $18+15-6=27$

$$P(\text{either})=\frac{27}{40}$$

$$\text{neither}=\frac{13}{40}.$$

Example 3.3.2

Fair die: A={1,2}, B={5,6}. Find the probability of A or B.

Solution:

Disjoint, $P(A \cup B) = \frac{2}{6} + \frac{2}{6} = \frac{2}{3}$.

Example 3.3.3 • Students' activities

How many students selected at least one activity?

Region	Students
Soccer only	12
Both	6
Basketball only	9
Neither	13

Illustrative classroom dataset

Example 3.3.3 • Solution

How many students selected at least one activity?

Solution:

12 + 6 + 9 = 27 of 40.

Individual practice

1. P(A)=.4, P(B)=.3, P(both)=.1.

2. P(A)=.72; find P(not A).

3. Are odd and even results disjoint on a die?

Group activity

Draw Venn regions for 50 students: 24 read, 20 draw, 10 both.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. P(union)=.6.

2. .28.

3. Yes.

Group solution

Solution:

Both 10
read only 14
draw only 10
neither 16.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Why subtract the intersection?

2. $P(A)=0.4$, $P(B)=0.5$ and $P(A \cap B)=0.2$. Find $P(A \cup B)$.

Show your reasoning. Submit before leaving.

Five-minute review

1. Why subtract the intersection?

2. P(A)=.4, P(B)=.3, P(both)=.1.

Write your answers before we discuss them.

Multiplication, trees and independence

What we will learn

Calculate joint probabilities with and without replacement.

Why does it matter?

Without replacement, the contents change after the first draw. The second probability must reflect what remains.

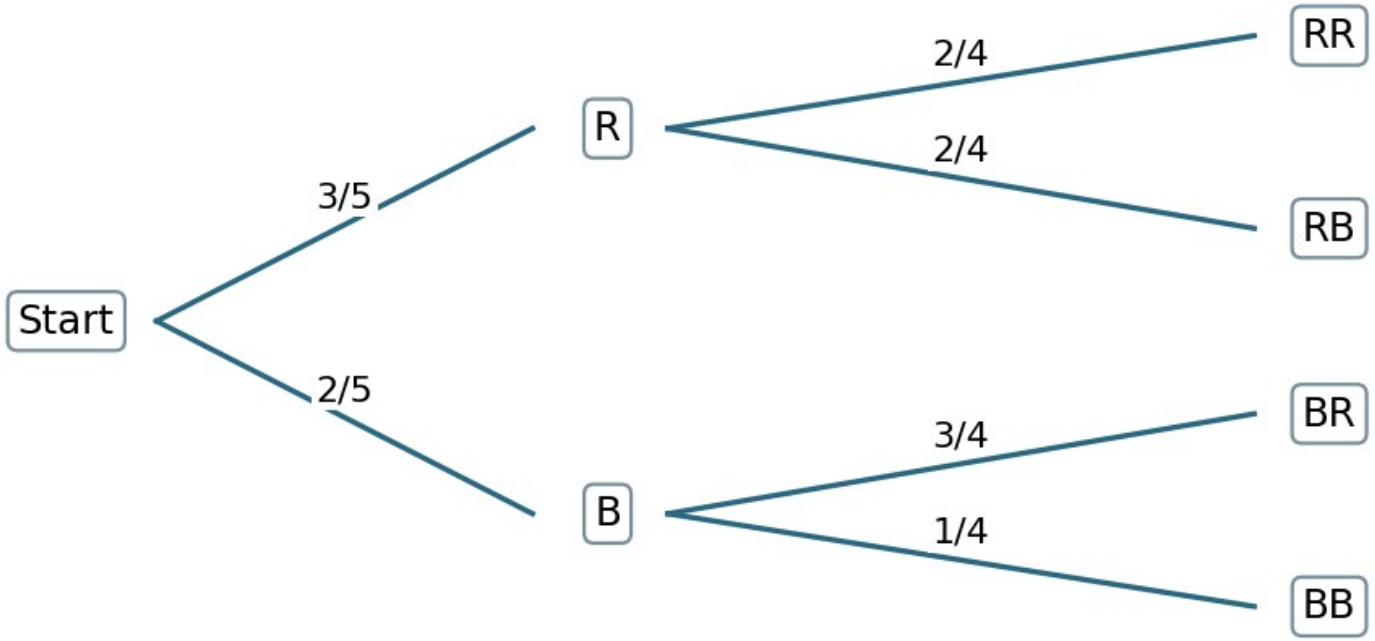
Definitions: Multiplication trees and independence

Independent events: knowing one occurred does not change the probability of the other.

Dependent events: one event can change the probability of another.

Tree diagram: branches show sequential outcomes and their conditional probabilities.

Illustration and interpretation



Notice the pattern

Without replacement, the contents change after the first draw. The second probability must reflect what remains.

Illustrative data / model

Formulas

$$P(A \cap B) = P(A)P(B \mid A)$$

$$P(A \cap B) = P(A)P(B) \quad \text{(independent)}$$

Important ideas and notation

General multiplication: $P(A \cap B) = P(A)P(B \mid A)$. Independent events satisfy $P(A \cap B) = P(A)P(B)$.

With replacement usually restores composition in random draws; without replacement generally changes it. Disjoint nonzero-probability events are not independent.

Tree diagrams multiply along branches and add probabilities of disjoint paths.

Example 3.4.1

Bag: 3 red, 2 blue. Draw twice without replacement. P(two red)?

Solution:

$$\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}.$$

Example 3.4.2

Same bag with replacement. P(red then blue)?

Solution:

$$\frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$$

Example 3.4.3 • Drawing colored counters

What is P(blue on second draw given red first)?

First draw	Remaining red	Remaining blue
Red	2	2
Blue	3	1

Illustrative classroom dataset

Example 3.4.3 • Solution

What is P(blue on second draw given red first)?

Solution:

$\frac{2}{4} = \frac{1}{2}$.

Individual practice

1. Independent events with probabilities .4 and .5. P(both)?

2. Bag 4 green, 6 yellow, without replacement. P(two green)?

3. Two fair coins, P(two heads)?

Group activity

Draw a tree for bag 2 red, 3 blue without replacement; find probability of different colors.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. .20.

2. $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$.

3. $\frac{1}{4}$.

Group solution

Solution:

RB $\frac{2}{5} \times \frac{3}{4} = .30$

BR $\frac{3}{5} \times \frac{2}{4} = .30$

sum .60.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Does mutually exclusive mean independent?

2. A bag contains 4 green and 2 yellow counters. Draw two without replacement. Find P(two yellow).

Show your reasoning. Submit before leaving.

Quick review — 5 minutes

1. A fair die is rolled. Find the probability of rolling an even number.

2. What is the difference between independent and mutually exclusive events?

Lesson 3.5: Conditional probability

What we will learn

- Interpret the phrase “given that.”
- Calculate conditional probabilities from a two-way table.
- Explain why reversing the condition may change the answer.

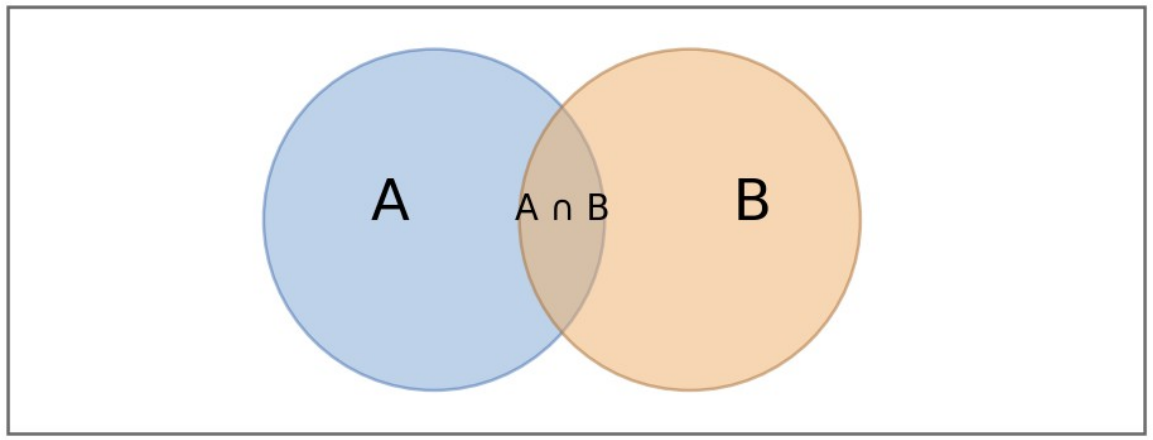
Why does it matter?

A library surveys students about whether they use digital or printed books. Does the preference change when we look at only one grade?

Definition: Conditional probability

Conditional probability is the probability that event A occurs when we know that event B has occurred.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0$$



Once B is known, the reference group is B, not the entire sample space.

Example 3.5.1: A two-way table

A survey of 80 students records their preferred reading format. One student is selected randomly.

Grade	Digital	Print	Total
9	18	22	40
10	26	14	40
Total	44	36	80

Find the probability that the student prefers digital books, given that the student is in Grade 10.

Example 3.5.1: Use the correct denominator

Let D = prefers digital books and G = student is in Grade 10.

Grade	Digital	Print	Total
10	26	14	40

$$P(D \mid G) = \frac{26}{40} = \frac{13}{20} = 0.65$$

65% of Grade 10 students in this survey prefer digital books.

Example 3.5.2: Reverse the condition

What is the probability that a randomly selected student is in Grade 10, given that the student prefers digital books?

Grade	Digital
9	18
10	26
Total	44

$$P(G \mid D) = \frac{26}{44} = \frac{13}{22}$$

$$\approx 0.591$$

The numerator is unchanged, but the denominator is different.

Example 3.5.3: Joint versus conditional probability

Using the same survey, find the probability that a student is in Grade 10 AND prefers digital books.

$$P(G \cap D) = \frac{26}{80} = 0.325$$

For a joint probability, divide by the total number of students. For a conditional probability, divide by the total of the given group.

Individual practice

1. Find P(Print | Grade 9).

2. Find P(Grade 9 | Print).

3. Find P(Grade 9 and Print).

Group activity

In a sports club, 24 of 40 members play basketball. Of those 24, 10 also play soccer.

Find P(soccer | basketball). Explain why the reverse probability cannot be found from these facts alone.

Solutions

1. 22 out of 40 = 0.55.

2. 22 out of 36 ≈ 0.611.

3. 22 out of 80 = 0.275.

Group: 10 out of 24 ≈ 0.417. The total number of soccer players is unknown.

Exit ticket — 5 minutes

1. What does the vertical bar in P(A | B) mean?

2. In the survey, find P(Print | Grade 10). Explain your denominator.

Five-minute review

1. What is the denominator in a conditional probability?

2. Find P(bus) in the table.

Write your answers before we discuss them.

Total probability, Bayes and at least one

What we will learn

Combine paths, update probabilities and use complements.

Why does it matter?

Even a reasonably accurate check can flag many good items when the defect is uncommon. Compare the correct conditional probabilities.

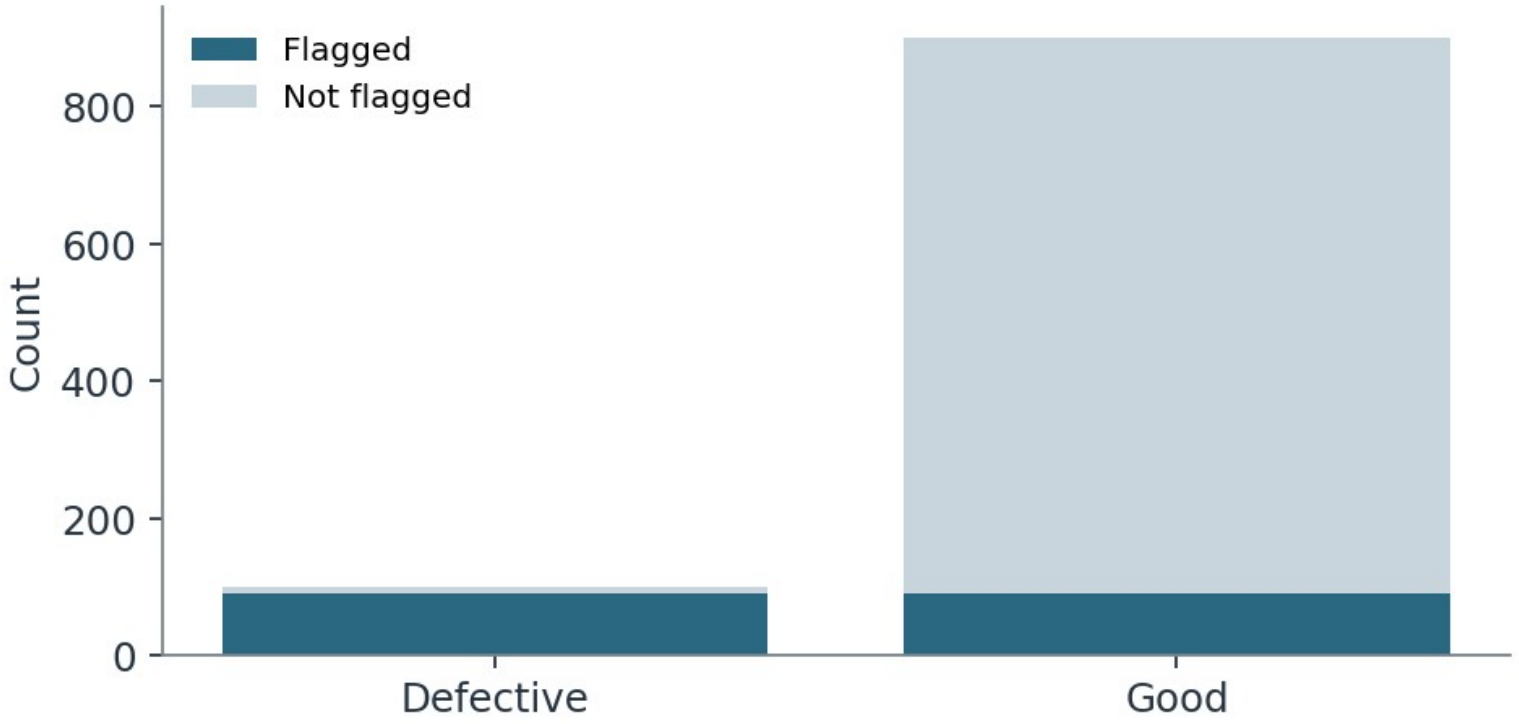
Definitions: Total probability bayes and at least one

Law of total probability: add weighted conditional probabilities across disjoint cases that cover the sample space.

Bayes' rule: reverses the direction of conditioning using joint and marginal probabilities.

At least one: one or more successes, the complement of zero successes.

Illustration and interpretation



Notice the pattern

Even a reasonably accurate check can flag many good items when the defect is uncommon. Compare the correct conditional probabilities.

Illustrative data / model

Formulas

$$P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c)$$

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$$

Important ideas and notation

Partition S into disjoint exhaustive groups B_i. $P(A)=\Sigma P(A \mid B_i)P(B_i)$.

Bayes: $P(B \mid A)=P(A \mid B)P(B)/P(A)$. A rare condition can remain unlikely after a positive screen.

P(at least one)=1−P(none). For independent identical trials, $P(\text{none})=(1-p)^n$.

Example 3.6.1

Two equally likely boxes: A has 3 red of 4, B has 1 red of 4. P(A | red)?

Solution:

$P(\text{red}) = .5(.75) + .5(.25) = .5$

$$P(A|\text{red}) = \frac{.375}{.5} = .75.$$

Example 3.6.2

Three independent attempts, success .2 each.

Solution:

$P(\text{at least one}) = 1 - .8^3 = .488.$

Example 3.6.3 • Interpreting a screening result

Of the flagged items, what fraction is defective?

Item	Flagged	Not flagged
Defective	90	10
Good	90	810

Illustrative classroom dataset

Example 3.6.3 • Solution

Of the flagged items, what fraction is defective?

Solution:

$$\frac{90}{90+90} = \frac{1}{2}.$$

Individual practice

1. Fair coin 4 tosses, at least one head.

2. Two equally likely machines with defect rates .02 and .06. Overall rate?

3. Given defective in previous question, probability from second machine?

Group activity

A hypothetical screen: 100 true cases in 1000; 90 test positive; 90 of 900 noncases test positive. Find P(case | positive).

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. $\frac{15}{16}$.

2. .04.

3. .5(.06)/.04=.75.

Group solution

Solution:

$\frac{90}{90+90} = .50$. Compare with sensitivity .90.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Why is at least one easier via complement?

2. Three independent trials each have success probability 0.3. Find P(at least one success).

Show your reasoning. Submit before leaving.

Five-minute review

1. Why is at least one easier via complement?

2. Fair coin 4 tosses, at least one head.

Write your answers before we discuss them.

Counting rules and permutations

What we will learn

Use addition and multiplication counting rules and permutations.

Why does it matter?

A tree or organized table prevents missed combinations. Independent choices multiply the number of possible outcomes.

Definitions: Counting rules and permutations

Fundamental counting principle: multiply the number of choices at successive stages.

Factorial $n!$: the product $n(n-1)\dots1$, with $0!=1$.

Permutation: an ordered arrangement of distinct selected objects.

Formulas

$$P(n, r) = \frac{n!}{(n - r)!}$$

Important ideas and notation

Addition principle counts disjoint alternatives; multiplication principle counts sequential choices. $n! = n(n-1)\ldots1$, and $0!=1$.

Permutations count ordered selections without replacement: $nPr=n!/(n-r)!$. If repetition is allowed, n choices per position gives n^r.

Decide whether order matters and whether repetition is allowed before choosing a formula.

Example 3.7.1

Choose one of 4 shirts and one of 3 pants. How many different outfits can be made?

Solution:

$4 \times 3 = 12$ outfits.

Example 3.7.2

Select president and vice president from 7 students, distinct people. How many ordered assignments are possible?

Solution:

$7P2 = 7 \times 6 = 42$.

Example 3.7.3 • Choosing a school outfit

How many outfits are possible?

Shirt	Pants choices
Blue	Black, gray, tan
White	Black, gray, tan
Green	Black, gray, tan
Red	Black, gray, tan

Illustrative classroom dataset

Example 3.7.3 • Solution

How many outfits are possible?

Solution:

$4 \times 3 = 12$.

Individual practice

1. Arrange 5 distinct books on a shelf.

2. How many 3-digit PINs (0–9, repetition allowed, leading zero allowed)?

3. Choose one of 4 teas or 3 juices.

Group activity

Compare ordered codes from 5 letters using 3 slots with and without repetition.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. $5! = 120$.

2. $10^3 = 1000$.

3. $4 + 3 = 7$ choices.

Group solution

Solution:

125 vs 5×4×3=60.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. When is nPr appropriate?

2. There are 6 students for captain and assistant captain. How many assignments are possible?

Show your reasoning. Submit before leaving.

Five-minute review

1. When is nPr appropriate?

2. Arrange 5 distinct books on a shelf.

Write your answers before we discuss them.

Combinations and counting probabilities

What we will learn

Use combinations and calculate probabilities using counts.

Why does it matter?

Order matters when assigning positions, but it does not matter when selecting an unordered team.

Definitions: Combinations and counting probabilities

Combination: an unordered selection of distinct objects.

Binomial coefficient nCr: the number of r-element subsets of n distinct objects.

Formulas

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Important ideas and notation

Combinations count unordered selections without replacement: $nCr = n!/[r!(n-r)!]$. Divide permutations by r! because order does not distinguish groups.

For equally likely groups, probability = favorable groups/total groups. Use products for separate subgroup choices.

Distinguish selecting a team from ranking its members.

Example 3.8.1

Choose 3 students from 8. How many groups can be chosen?

Solution:

$8C3=56$.

Example 3.8.2

Choose 2 red from 5 and 1 blue from 4. How many selections are possible?

Solution:

$5C2 \times 4C1 = 10 \times 4 = 40.$

Example 3.8.3 • Selecting a project team

Which choice uses combinations?

Choice	Order matters?
Choose three members	No
Choose president and secretary	Yes
Arrange presentation order	Yes

Illustrative classroom dataset

Example 3.8.3 • Solution

Which choice uses combinations?

Solution:

Selecting three members without roles.

Individual practice

1. Choose 2 from 10.

2. How many ways to award gold and silver among 10?

3. Draw 2 from 4 red and 6 blue. P(both red)?

Group activity

A committee of 3 from 6 juniors and 4 seniors is chosen randomly. P(exactly 2 juniors)?

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. 45.

2. $10P2=90$.

3. $4C2/10C2=\frac{6}{45}=\frac{2}{15}$.

Group solution

Solution:

$$6C2 \times 4C1 / 10C3 = \frac{60}{120} = .5.$$

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Why divide by r! in combinations?

2. How many committees of 2 can be chosen from 9 students?

Show your reasoning. Submit before leaving.

Five-minute review

1. Why divide by r! in combinations?

2. Choose 2 from 10.

Write your answers before we discuss them.

Counting by cases and restrictions

What we will learn

Use addition and multiplication principles to count mutually exclusive cases and restricted outcomes.

Why does it matter?

Count separate menu categories, then add only if the cases cannot overlap.

Definitions: Counting by cases and restrictions

Disjoint cases: separate groups of outcomes that do not overlap.

Restriction: a condition that limits the available choices at one or more stages.

Important ideas and notation

Addition principle: add the counts for disjoint alternatives. If cases overlap, subtract the overlap or redefine cases.

Multiplication principle: multiply choices for successive stages when each stage has the stated number of options. Use separate cases if later choices vary.

Using the method correctly

A tree diagram can track restrictions, such as no repeated digits or a first digit that cannot be zero. Probability = favorable count / total count only when outcomes are equally likely.

Example 3.9.1

A café has 4 sandwiches with 3 sides or 2 salads with 3 sides. Count meals.

Solution:

Sandwich meals = 12

salad meals = 6. Disjoint cases total 18.

Example 3.9.2

Count three-digit codes using digits 0–4, no repeats, first digit not zero.

Solution:

$4 \times 4 \times 3 = 48$. First digit has 4 choices, second 4, third 3.

Example 3.9.3 • Counting a lunch menu

How many meals have one main item and one side?

Meal type	Main choices	Side choices
Sandwich	4	3
Salad	2	3

Illustrative classroom dataset

Example 3.9.3 • Solution

How many meals have one main item and one side?

Solution:

$4 \times 3 + 2 \times 3 = 18$.

Individual practice

1. Choose one of 5 novels or one of 4 biographies. How many choices?

2. A code uses a letter A–D followed by two distinct digits from 0–4. Count codes.

3. Of the 48 restricted three-digit codes in Example 2, how many start with 2?

Group activity

A club can choose a project in art (3 projects, 2 presentation formats) or science (4 projects, 3 formats). Draw a tree or table, count outcomes and find the chance of a science outcome if all outcomes are equally likely.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. 9.

2. $4 \times 5 \times 4 = 80$.

3. $1 \times 4 \times 3 = 12$; probability $\frac{12}{48} = \frac{1}{4}$ if uniformly selected.

Group solution

Solution:

Art 6

science 12

total 18

$P(\text{science}) = \frac{12}{18} = \frac{2}{3}$. Choosing categories first with equal probability would be a different model.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. When is it valid to add counts from two cases directly?

2. How many two-digit numbers can be made from digits 1,2,3,4 without repetition?

Show your reasoning. Submit before leaving.

Five-minute review

1. When is it valid to add counts from two cases directly?

2. Choose one of 5 novels or one of 4 biographies. How many choices?

Write your answers before we discuss them.

Spreadsheet probability simulation

What we will learn

Design a reproducible simulation and compare experimental and theoretical probability.

Why does it matter?

Use random 0s and 1s to model fair coins. A success indicator equals 1 only if both cells contain 1.

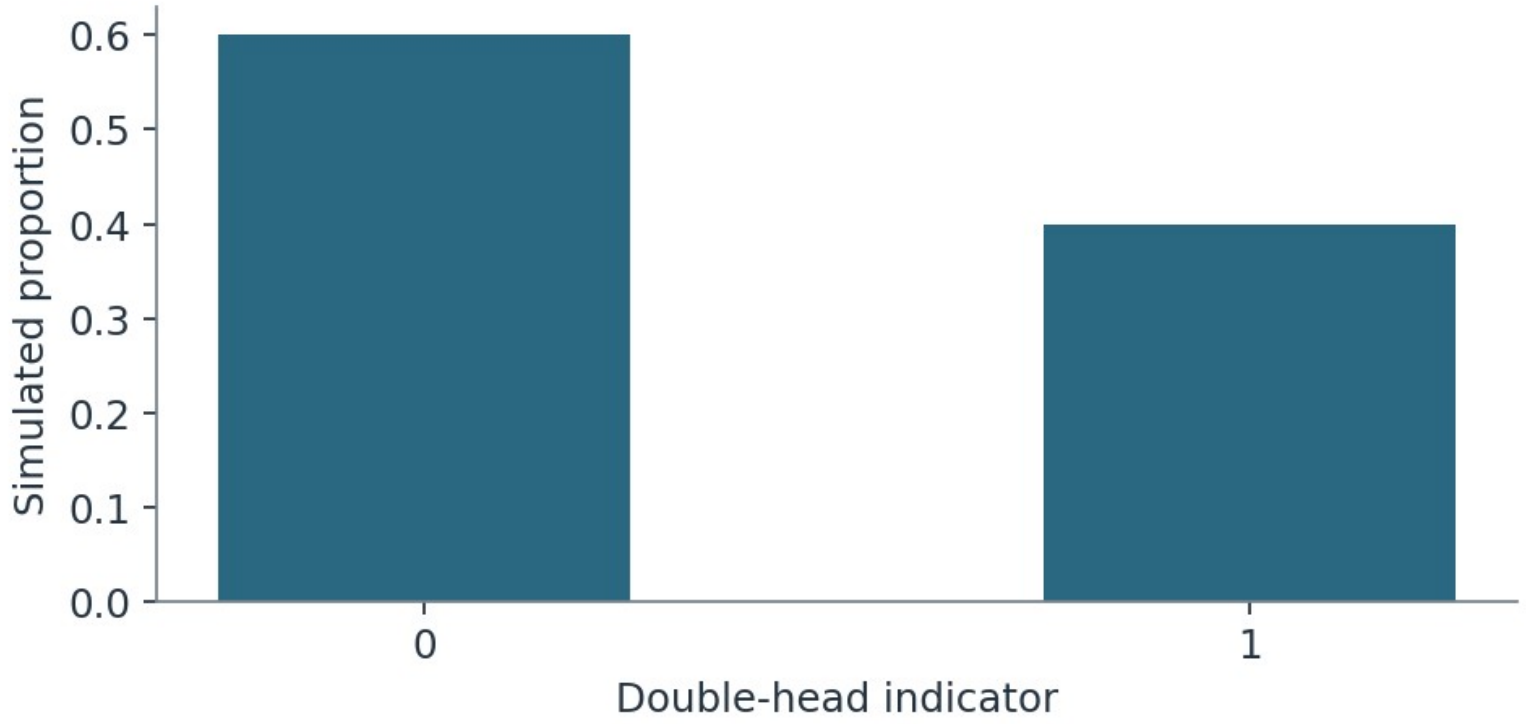
Definitions: Spreadsheet probability simulation

Random-number generator: a tool producing simulated random values.

Trial: one repetition of a specified experiment.

Simulation estimate: the observed fraction of successful trials.

Illustration and interpretation



Notice the pattern

Use random 0s and 1s to model fair coins. A success indicator equals 1 only if both cells contain 1.

Illustrative data / model

Formulas

$$\widehat{P}(A) = \frac{\text{Simulated successes}}{\text{Trials}}$$

Important ideas and notation

A simulation uses a random process to imitate a model. Specify outcomes, probabilities, a trial, number of trials and success condition.

In a spreadsheet, =RANDBETWEEN(0,1) represents a fair toss. In A2 and B2 generate two tosses; C2 = IF(A2+B2=2,1,0). Fill to row 101.

Using the method correctly

Estimate probability using =AVERAGE(C2:C101). Recalculation changes random results; paste values to preserve a run. More trials generally improve stability, not guarantee exact equality.

Example 3.10.1

A simulation of 100 two-coin trials records 28 double-heads. Estimate and compare with theory.

Solution:

Estimate .28

theory .25

difference .03 (3 percentage points).

Example 3.10.2

A six-sided fair die simulation records 164 sixes in 1000 rolls. Estimate P(6).

Solution:

$$\frac{164}{1000} = .164$$

theoretical $\frac{1}{6} \approx .1667$.

Example 3.10.3 • A spreadsheet simulation

What is the simulated double-head proportion in these five trials?

Trial	Coin 1	Coin 2	Two heads
1	1	0	0
2	1	1	1
3	0	0	0
4	1	1	1
5	0	1	0

Illustrative classroom dataset

Example 3.10.3 • Solution

What is the simulated double-head proportion in these five trials?

Solution:

$\frac{2}{5} = 0.40$

a small run can differ from 0.25.

Individual practice

1. Use 0, 1, 2, 3, 4 to simulate a 40% event. Which outcomes count as success?

2. In 200 trials, 74 succeed. Find experimental probability.

3. A 50-trial run differs more from theory than a 500-trial run. Must this always occur?

Group activity

In pairs, simulate two fair coins for 100 trials using the given spreadsheet formulas. Combine class results, calculate the pooled probability and compare with 1/4.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. Any two of the five equally likely outcomes, e.g. 0 and 1.

2. $\frac{74}{200} = .37.$

3. No. Longer runs tend to stabilize, but any individual run can deviate.

Group solution

Solution:

Individual and pooled results vary. Pooled estimate = total double-heads / total trials

compare with .25 and discuss random variation.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. What must be specified before running a probability simulation?

2. A spreadsheet simulation of 500 trials gives 126 successes. Estimate success probability.

Show your reasoning. Submit before leaving.

Five-minute review

1. What must be specified before running a probability simulation?

2. Use 0, 1, 2, 3, 4 to simulate a 40% event. Which outcomes count as success?

Write your answers before we discuss them.

Fair decisions using random numbers

What we will learn

Construct a fair random selection method and recognize bias in number assignments.

Why does it matter?

A fair decision requires equal selection probabilities, not simply using a random number generator.

Definitions: Fair decisions using random numbers

Fair selection: each eligible participant has equal selection probability.

Rejection sampling (in this lesson): discard unused random outcomes and draw again.

Formulas

$$P(A) = \frac{\text{Favorable outcomes}}{\text{Equally likely outcomes}}$$

Important ideas and notation

A fair selection gives each eligible person or team the intended equal probability. Assign equally many equally likely random outcomes to each.

If the number of possible random outcomes is not divisible by the number of choices, reject excess outcomes and repeat. Avoid uneven grouping or rounding.

Using the method correctly

To choose without replacement, remove the selected item before the next draw. Use shuffled lists or distinct random selections; preserve the recorded results for transparency.

Example 3.11.1

Three teams use a fair die. Assign 1–2 to A, 3–4 to B, 5–6 to C. Is the rule fair?

Solution:

Yes. Each receives 2 of 6 equally likely outcomes, probability $\frac{1}{3}$.

Example 3.11.2

Four students use a die. Assign 1–4 to students and reroll 5 or 6. Is this fair?

Solution:

Yes. Conditional on an accepted roll, each student has probability $\frac{1}{4}$.

Example 3.11.3 • Fairly choosing a team

Why is this assignment fair for three teams?

Die outcome	Team
1 or 2	A
3 or 4	B
5 or 6	C

Illustrative classroom dataset

Example 3.11.3 • Solution

Why is this assignment fair for three teams?

Solution:

Each gets two equally likely outcomes, probability $\frac{2}{6} = \frac{1}{3}$.

Individual practice

1. Five students use digits 0–9, assigning two digits to each. Find each selection probability.

2. A die assigns 1–3 to A, 4–5 to B, and 6 to C. Is it fair?

3. Seven students use digits 0–9. Design a fair rule.

Group activity

Design a transparent method for randomly choosing two of six project teams without replacement. List possible first and second selections and explain why every unordered pair is equally likely.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. $\frac{2}{10} = \frac{1}{5}$.

2. No; probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

3. Assign 0–6 to students; reject 7–9 and redraw.

Group solution

Solution:

Roll a die for first team

reroll its number for second. Each ordered pair has probability $\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$

each unordered pair $\frac{2}{30} = \frac{1}{15}$.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Why should rejected random numbers be redrawn rather than assigned to one participant?

2. Five students use a fair die, with 1–5 assigned and 6 discarded. What is each student's probability after an accepted roll?

Show your reasoning. Submit before leaving.

Five-minute review

1. Why should rejected random numbers be redrawn rather than assigned to one participant?

2. Five students use digits 0–9, assigning two digits to each. Find each selection probability.

Write your answers before we discuss them.

Conditional risk and two-way tables

What we will learn

Use conditional relative frequencies to evaluate risk and make evidence-based decisions.

Why does it matter?

A flag is evidence, not certainty. A two-way table distinguishes the chance of being flagged if defective from the chance of a defect if flagged.

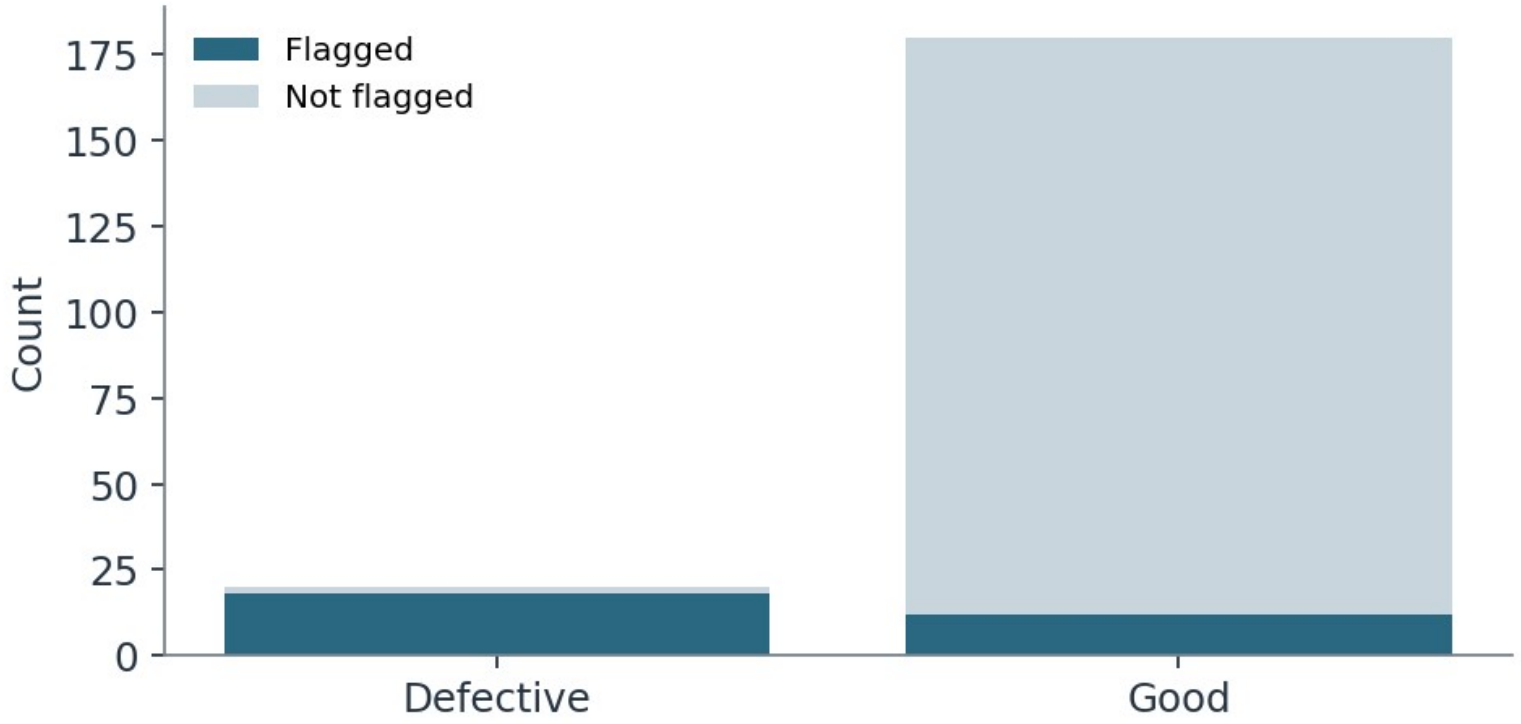
Definitions: Conditional risk and two way tables

Sensitivity: probability of a positive flag given that an item really is defective.

Positive predictive value: probability an item is defective given a positive flag.

Base rate: the proportion of defective items in the population or sample.

Illustration and interpretation



Notice the pattern

A flag is evidence, not certainty. A two-way table distinguishes the chance of being flagged if defective from the chance of a defect if flagged.

Illustrative data / model

Formulas

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Important ideas and notation

Conditional probability P(A | B) = count(A and B)/count(B) in a frequency table with equally likely selection. The condition determines the denominator.

Reverse conditions generally give different results. P(flag | defective) is not P(defective | flag). Joint probability uses the grand total.

Using the method correctly

Compare conditional rates rather than raw counts when groups have different sizes. A probability can inform a decision, but costs and limitations also matter.

Example 3.12.1

Of 200 items, 20 are defective. A check flags 18 defective and 12 good items. Find P(flag | defective).

Solution:

$\frac{18}{20} = .90$. The check flags 90% of defective items.

Example 3.12.2

Using the same data, find P(defective | flag).

Solution:

$\frac{18}{18+12}$ = .60. Only 60% of flagged items are defective.

Example 3.12.3 • Quality inspection

Find P(defective given flagged).

Item	Flagged	Not flagged
Defective	18	2
Good	12	168

Illustrative classroom dataset

Example 3.12.3 • Solution

Find P(defective given flagged).

Solution:

$\frac{18}{30} = 0.60$, not 0.90.

Individual practice

1. Of 80 morning riders, 12 are late; of 40 evening riders, 8 are late. Find both late rates.

2. In the quality example, find P(good | not flagged).

3. Of 120 riders above, choose one. Find P(evening and late).

Group activity

Create the full 2×2 table for the quality-check example. Compare the chance a flagged item is defective with the overall defective rate. Recommend a follow-up decision.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. Morning $\frac{12}{80} = .15$; evening $\frac{8}{40} = .20$.

2. $\frac{168}{170} \approx .9882$.

3. $\frac{8}{120} = \frac{1}{15} \approx .0667$.

Group solution

Solution:

Defective: flagged 18, not 2. Good: flagged 12, not 168. Totals 30, 170, 200. Flagged risk .60 versus overall .10

inspect flagged items further rather than assuming all are defective.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. What is the denominator in P(defective | flag)?

2. A check flags 15 defective and 5 nondefective items. Among flagged items, what proportion are defective?

Show your reasoning. Submit before leaving.