Five-minute review

1. What is the denominator in P(defective | flag)?

2. Of 80 morning riders, 12 are late; of 40 evening riders, 8 are late. Find both late rates.

Write your answers before we discuss them.

Random variables and discrete distributions

What we will learn

Build and validate a discrete probability distribution.

Why does it matter?

A random variable assigns a number to each outcome. Several outcomes can have the same value.

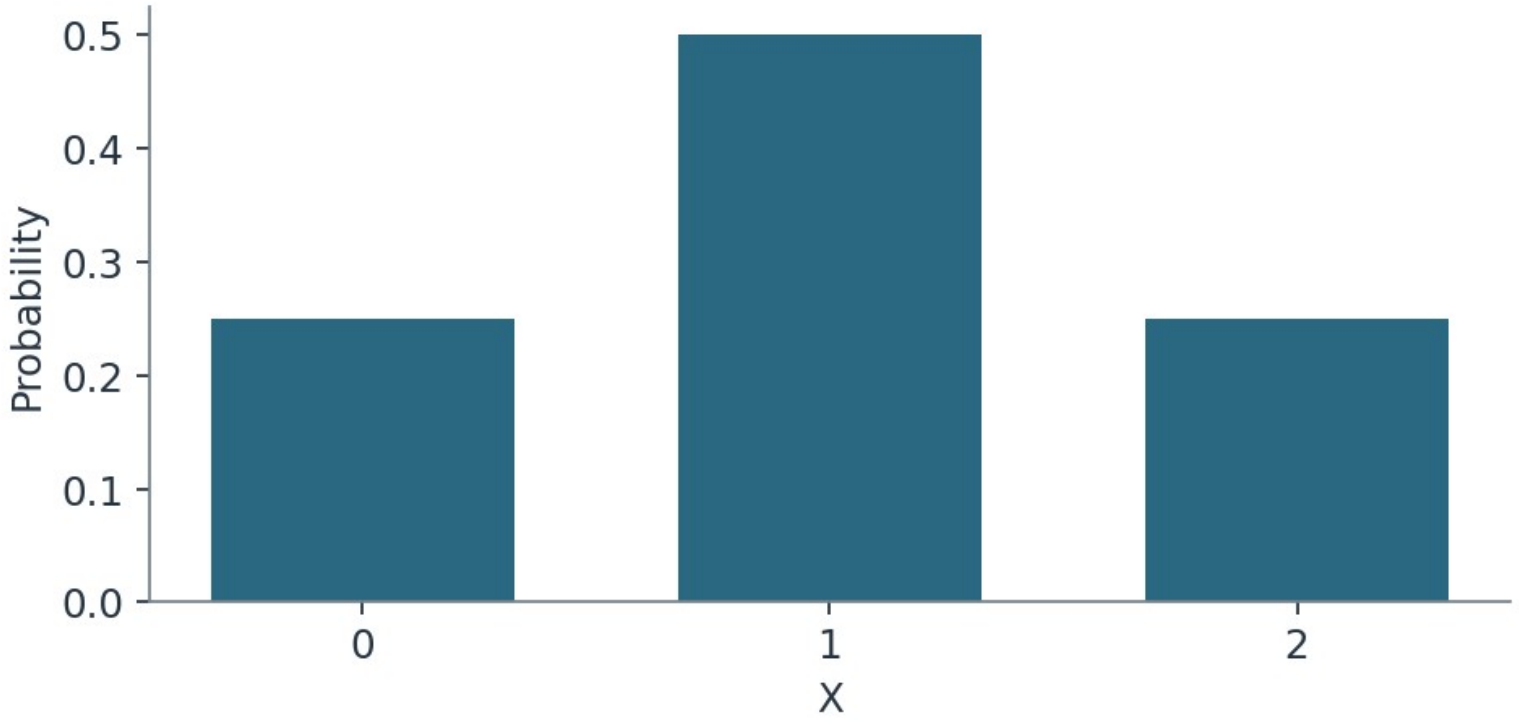
Definitions: Random variables and discrete distributions

Random variable: a numerical function of the outcome of a random experiment.

Discrete random variable: takes a finite or countable set of values.

Probability distribution: specifies the probability of each possible value (for a discrete variable).

Illustration and interpretation



Notice the pattern

A random variable assigns a number to each outcome. Several outcomes can have the same value.

Illustrative data / model

Formulas

$$0 \le P(X = x) \le 1, \quad \sum_{x} P(X = x) = 1$$

Important ideas and notation

A random variable X assigns a numerical value to each outcome. Discrete variables take countable values; continuous variables are measured on intervals.

A discrete probability distribution lists possible x and P(X=x). All probabilities are between 0 and 1 and sum to 1.

Use a table, graph or formula. For an event involving multiple x values, add their probabilities.

Example 4.1.1

Two independent fair tosses. X=head count. Construct the probability distribution of X.

Solution:

$TT \rightarrow 0$, $HT/TH \rightarrow 1$, $HH \rightarrow 2$

$P(0)=.25$, $P(1)=.50$, $P(2)=.25$.

Example 4.1.2

A proposed distribution has probabilities .2, .3, .6. Is the distribution valid? Explain.

Solution:

Sum 1.1, invalid.

Example 4.1.3 • Number of heads in two tosses

Why is one head more likely than two heads?

Heads x	Outcomes	P(X=x)
0	TT	0.25
1	HT, TH	0.50
2	HH	0.25

Illustrative classroom dataset

Example 4.1.3 • Solution

Why is one head more likely than two heads?

Solution:

Two equally likely outcomes give one head, only one gives two.

Individual practice

1. X values 0,1,2 with probabilities .1,.6,?.

2. Find $P(X \geq 1)$ for the two-coin example.

3. Can a random variable take negative values?

Group activity

Construct a table and graph for three independent fair coin tosses.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. Missing .3.

2. .75.

3. Yes, e.g., net change in points; probabilities cannot be negative.

Group solution

Solution:

$x=0,1,2,3$

probabilities $\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8}$.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. What two rules validate a probability distribution?

2. X takes values 0,1,2 with probabilities 0.2,0.5 and p. Find p.

Show your reasoning. Submit before leaving.

Five-minute review

1. What two rules validate a probability distribution?

2. X values 0,1,2 with probabilities .1,.6,?.

Write your answers before we discuss them.

Expected value and variance

What we will learn

Calculate expected value and spread for a discrete distribution.

Why does it matter?

Expected value is a probability-weighted average, not necessarily a result that occurs in any single round.

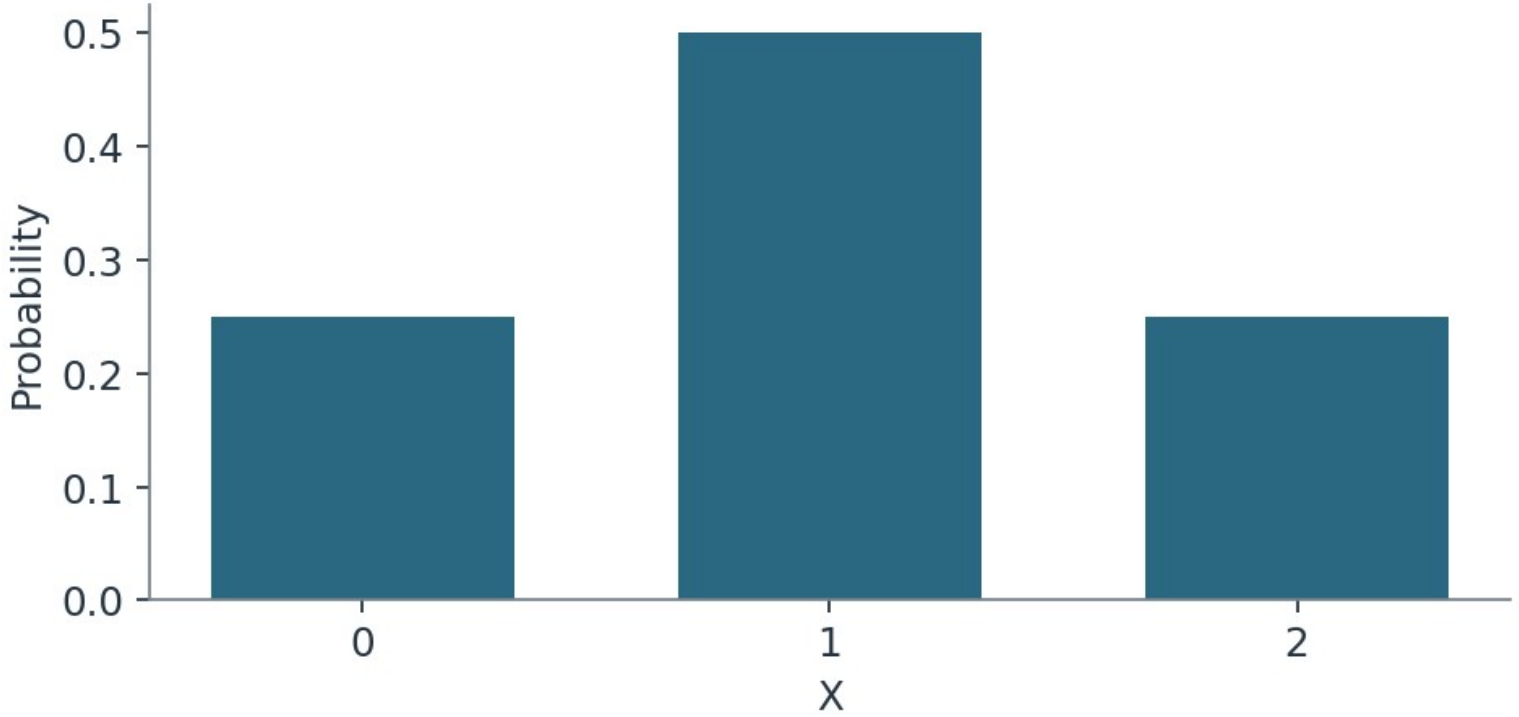
Definitions: Expected value and variance

Expected value: the probability-weighted average of possible values.

Variance of a random variable: the probability-weighted average squared distance from its expected value.

Standard deviation: square root of that variance.

Illustration and interpretation



Notice the pattern

Expected value is a probability-weighted average, not necessarily a result that occurs in any single round.

Illustrative data / model

Formulas

$$\mu = E(X) = \sum_{x} xP(X = x)$$

$$\sigma^2 = \sum_{x} (x - \mu)^2 P(X = x), \quad \sigma = \sqrt{\sigma^2}$$

Important ideas and notation

Expected value $\mu = E(X) = \Sigma x P(x)$, the long-run average under the model. It need not be an attainable individual result.

Variance $\sigma^2 = \Sigma (x-\mu)^2 P(x) = \Sigma x^2 P(x) - \mu^2$; $\sigma = \sqrt{\sigma^2}$. These are population/model quantities, not sample estimates.

Expected net value accounts for all costs or gains. A fair points game has expected net gain zero.

Example 4.2.1

X=0,1,2 with p=.25,.5,.25. Calculate the expected value, variance and standard deviation.

Solution:

$E(X)=1$

$E(X^2)=1.5$

variance=.5

SD≈.707.

Example 4.2.2

A game awards 6 points with probability .25 and loses 2 otherwise. Calculate the expected net points and determine whether the game is fair.

Solution:

$$E(X)=6(.25)-2(.75)=0$$

fair in expected points.

Example 4.2.3 • Long-run points per round

What is the expected number of points?

Points	Probability
0	0.25
1	0.50
2	0.25

Illustrative classroom dataset

Example 4.2.3 • Solution

What is the expected number of points?

Solution:

$0(.25)+1(.50)+2(.25) = 1.$

Individual practice

1. X=0,4 with probabilities .5,.5. Find mean.

2. For same distribution, variance and SD.

3. Fair die expected value?

Group activity

Compare A: certain 3 points; B: 0 or 6 equally likely.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. 2.

2. 4 and 2.

3. 3.5.

Group solution

Solution:

Both expected 3

A variance 0

B variance 9. B is riskier.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. Does expected value guarantee what happens next?

2. X takes values 0 and 4 with equal probability. Find E(X) and SD(X).

Show your reasoning. Submit before leaving.

Five-minute review

1. Does expected value guarantee what happens next?

2. X=0,4 with probabilities .5,.5. Find mean.

Write your answers before we discuss them.

Binomial probabilities

What we will learn

Check binomial conditions and calculate exact probabilities.

Why does it matter?

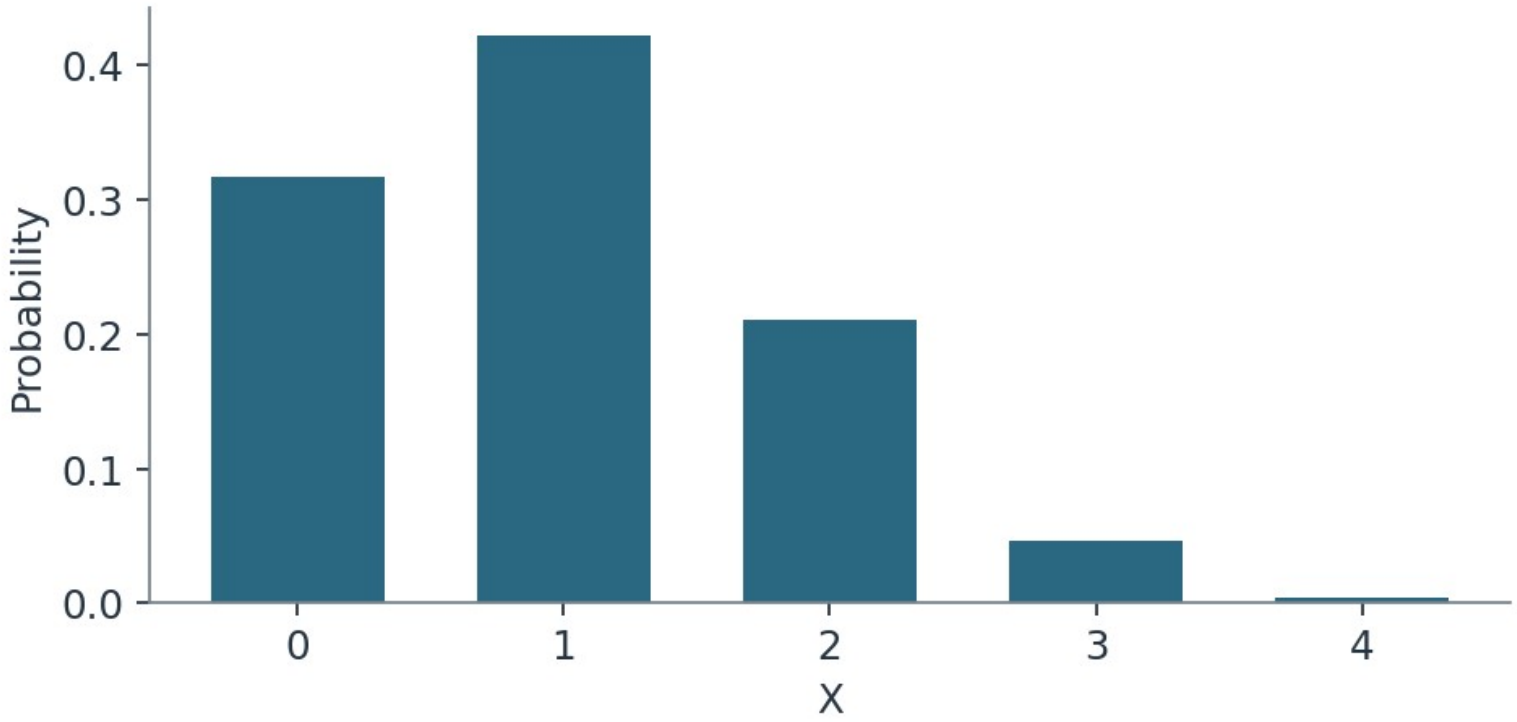
With independent random guesses and the same chance of success on each question, a binomial model counts correct responses.

Definitions: Binomial probabilities

Bernoulli trial: a trial with two outcomes, labeled success and failure.

Binomial random variable: the number of successes in a fixed number of independent Bernoulli trials with the same p.

Illustration and interpretation



Notice the pattern

With independent random guesses and the same chance of success on each question, a binomial model counts correct responses.

Illustrative data / model

Formulas

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n - k}$$

Important ideas and notation

Binomial conditions: fixed n, two outcomes per trial, independent trials, constant success probability p. Let X count successes.

$P(X=k)=nCk\ p^k(1-p)^{n-k}$ for k=0,…,n. Distinguish exactly, at most and at least.

Use binomial formula or R dbinom(k,n,p). If sampling without replacement substantially changes probabilities, binomial is not appropriate.

Example 4.3.1

Four independent questions, p=.25. P(exactly 2 correct).

Solution:

$4C2(.25)^2(.75)^2=.2109375.$

Example 4.3.2

Five independent attempts, p=.4. P(no success).

Solution:

$.6^5=.07776.$

Example 4.3.3 • A four-question quiz

What does the factor C(4,2) count?

Quantity	Value
Questions n	4
Chance correct p	0.25
Target k	2

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Example 4.3.3 • Solution

What does the factor C(4,2) count?

Solution:

The six possible placements of two correct answers among four questions.

Individual practice

1. n=3,p=.5; P(X=2).

2. n=4,p=.2; P(X=0).

3. Draw 4 from a bag of 10 without replacement. Binomial?

Group activity

Compute $P(X=1)$ and $P(X \geq 1)$ for $n=4, p=.3$.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. $\frac{3}{8} = .375$.

2. .4096.

3. Generally no, trials are dependent and p changes.

Group solution

Solution:

$4(.3)(.7^3)=.4116$

$1-.7^4=.7599.$

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. What does nCk count?

2. Three independent attempts have success probability 0.5. Find P(exactly 2 successes).

Show your reasoning. Submit before leaving.

Five-minute review

1. What does nCk count?

2. n=3,p=.5; P(X=2).

Write your answers before we discuss them.

Binomial cumulative probabilities and moments

What we will learn

Compute cumulative probabilities, expected count and binomial SD.

Why does it matter?

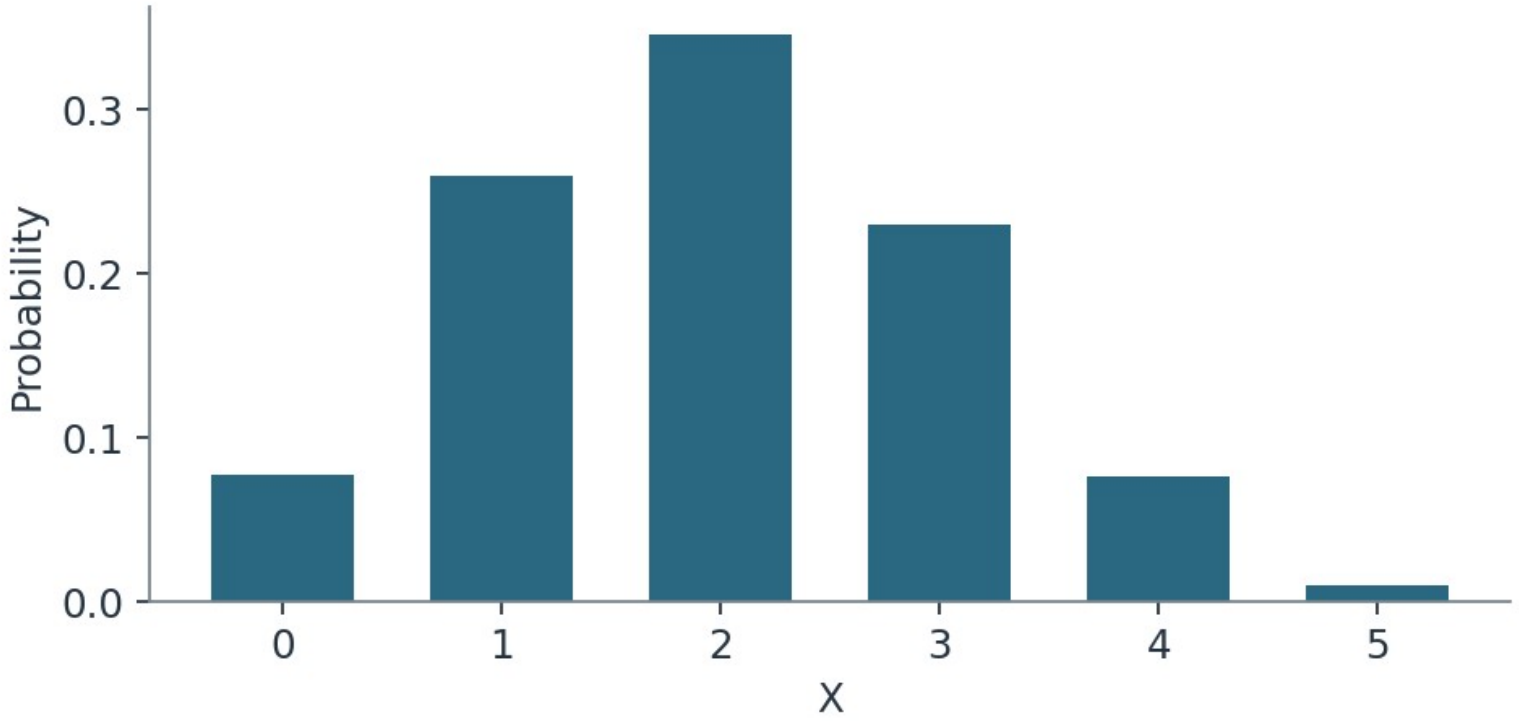
A binomial distribution can answer questions such as 'at most one' or 'at least one' by combining probabilities.

Definitions: Binomial cumulative probabilities and moments

Cumulative probability: $P(X \le k)$, the sum of probabilities for values no greater than k.

Binomial mean: expected count np; binomial variance: $np(1-p)$.

Illustration and interpretation



Notice the pattern

A binomial distribution can answer questions such as 'at most one' or 'at least one' by combining probabilities.

Illustrative data / model

Formulas

$$\mu = np, \quad \sigma = \sqrt{np(1-p)}$$

$$P(X \geq k) = 1 - P(X \leq k-1)$$

Important ideas and notation

For binomial X, $\mu=np$, $\sigma^2=np(1-p)$, $\sigma=\sqrt{[np(1-p)]}$.

$P(X\le k)=\Sigma$ from $j=0$ to k of $P(X=j)$. $P(X\ge k)=1-P(X\le k-1)$. Use R pbinom(k,n,p) for lower tail.

Report expected counts as long-run averages; choose correct tail and check whether a boundary is included.

Example 4.4.1

n=20,p=.25: mean and SD.

Solution:

$\mu = 5$

$\sigma = \sqrt{3.75} \approx 1.936.$

Example 4.4.2

$n=4, p=.5$: $P(X \le 1)$, $P(X \ge 2)$.

Solution:

$[1+4]/16=\frac{5}{16}$

complement $\frac{11}{16}$.

Example 4.4.3 • Quality checks in batches

Which outcomes are included in $X \le 1$?

Quantity	Value
Items inspected	5
Defect probability	0.4
Expected defects	2

Illustrative classroom dataset

Example 4.4.3 • Solution

Which outcomes are included in $X \leq 1$?

Solution:

$X=0$ and $X=1$ only.

Individual practice

1. $n=10, p=.6$; mean.

2. Same distribution, variance.

3. $n=3, p=.2$; $P(X \geq 1)$.

Group activity

For $n=5, p=.4$, find $P(X \leq 1)$ and expected count.

Show your calculation or graph. Write a sentence explaining the result in context.

Check your answers

1. 6.

2. 2.4.

3. $1-.8^3=.488$.

Group solution

Solution:

$.6^5+5(.4)(.6^4)=.33696$

expected 2.

Compare your conclusion with the question. Check your units and explain what the numbers represent.

Exit ticket • 5 minutes

1. What does at most 3 mean?

2. For X binomial with n=10 and p=0.3, find mean and SD.

Show your reasoning. Submit before leaving.